Regional Mathematical Olympiad – 2003

Time: 3 hours **7 December 2003**

1. Let ABC be a triangle in which $AB = AC$ and $\angle CAB = 90°$. Suppose M and N are points on the hypotenuse BC such that $BM^2 + CN^2 = MN^2$. Prove that $\angle MAN = 45°$.

2. If n is an integer greater than 7, prove that $\binom{n}{7} - \left[\frac{n}{7}\right]$ is divisible by 7.

   [Here $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects; also, for any real number x, $[x]$ denotes the greatest integer not exceeding x.]

3. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Prove that among the three numbers $a - ab, b - bc, c - ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.

4. Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:

   (i) $x \le y \le z$;

   (ii) $x + y + z \le 100$.

5. Suppose P is an interior point of a triangle ABC such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \quad \frac{d(B, CA)}{d(P, CA)}, \quad \frac{d(C, AB)}{d(P, AB)}$$

   are all equal. Find the common value of these ratios.

   [Here $d(X, YZ)$ denotes the perpendicular distance from a point X to the *line* YZ.]

6. Find all real numbers a for which the equation

$$x^2 + (a - 2)x + 1 = |x|$$

   has exactly three distinct real solutions in x.

7. Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

   (a) $A \cup B = X$;

   (b) $prod(A)$ is divisible by $prod(B)$, where for any finite set of numbers C, $prod(C)$ denotes the product of all numbers in C;

   (c) the quotient $prod(A)/prod(B)$ is as small as possible.