Regional Mathematical Olympiad – 2000

Time: 3 hours **3 December 2000**

1. Let AC be a line segment in the plane and B a point between A and C. Construct isosceles triangles PAB and QBC on one side of the segment AC such that $\angle APB = \angle BQC = 120^\circ$ and an isosceles triangle RAC on the other side of AC such that $\angle ARC = 120^\circ$. Show that PQR is an equilateral triangle.

2. Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$ for positive integers x and y.

3. Suppose $< x_1, x_2, \ldots, x_n, \ldots >$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq \cdots$, and for all n

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all k the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_k}{k} \leq 3.$$

4. All the 7-digit numbers containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once, and not divisible by 5, are arranged in increasing order. Find the 2000-th number in this list.

5. The internal bisector of angle A in a triangle ABC with $AC > AB$ meets the circumference Γ of the triangle in D. Join D to the centre O of the circle Γ and suppose DO meets AC in E, possibly when extended. Given that BE is perpendicular to AD, show that AO is parallel to BD.

6. (i) Consider two positive integers a and b which are such that $a^a b^b$ is divisible by 2000. What is the least possible value of the product ab?

   (ii) Consider two positive integers a and b which are such that $a^b b^a$ is divisible by 2000. What is the least possible value of the product ab?

7. Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.