

Regional Mathematical Olympiad-2011

Time: 3 hours

December 04, 2011

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions. Maximum marks: 100.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K . Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$. Prove that $\angle ABE = \angle CAD$. [12]
2. Let $(a_1, a_2, a_3, \dots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers $1, 2, 3, \dots, 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$. [19]
3. A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding l to n . Prove that $n - kl$ is a perfect square. [12]
4. Consider a 20-sided convex polygon K , with vertices A_1, A_2, \dots, A_{20} in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.) [19]
5. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B, \angle C$ with B_1 on AC and C_1 on AB . Let E, F be the feet of perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB . Prove that $AD = EF$. [19]
6. Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

[19]